Math 4300
Homework 9
Solutions

1 (a) Let A, B be distinct points in a Pasch geometry.

In HW 7 we showed that

$\overline{AB}$, $\text{int}(\overline{AB})$, $\overleftrightarrow{AB}$, $\overrightarrow{AB}$, and $\text{int}(\overrightarrow{AB})$

are all convex sets.

Let S be $\overline{AB}$, $\text{int}(\overline{AB})$, $\overleftrightarrow{AB}$, $\overrightarrow{AB}$, or $\text{int}(\overrightarrow{AB})$.

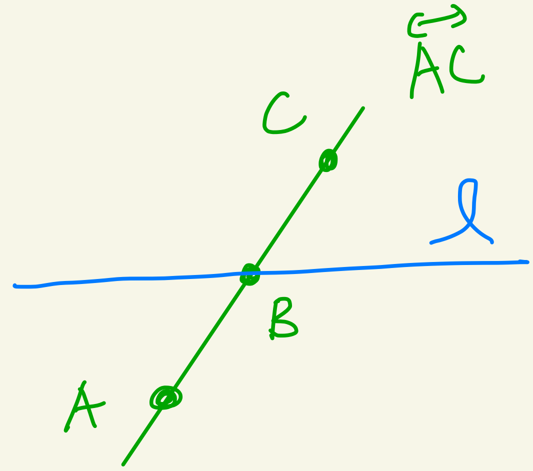
Then S is convex.

Thus, from a theorem in class, if $S \cap \ell = \phi$, then all the points of S lie on the same side of ℓ.

$\blacksquare$

1(b) Suppose we have a Pasch geometry.

Suppose $A-B-C$ and $\overleftrightarrow{AC} \cap \ell = \{B\}$.



Recall that

$$\operatorname{int}(\overrightarrow{BA}) = \overrightarrow{BA} - \{B\}$$

and

$$\operatorname{int}(\overline{BA}) = \overline{BA} - \{A, B\}$$

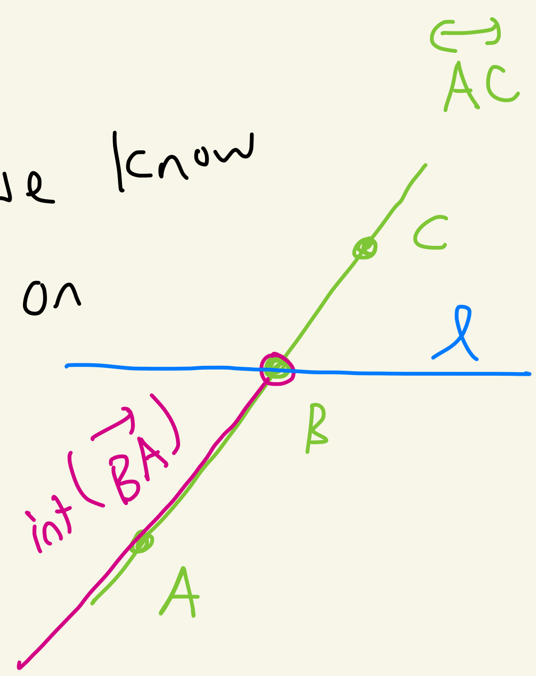
Since $A-B-C$ we know

$$\operatorname{int}(\overline{BA}) \subseteq \operatorname{int}(\overrightarrow{BA}) \subseteq \overrightarrow{BA} \subseteq \overleftrightarrow{BA} \subseteq \overleftrightarrow{AC}$$

Since $B \notin \operatorname{int}(\overrightarrow{BA})$ and $\operatorname{int}(\overrightarrow{BA}) \subseteq \overleftrightarrow{AC}$ and $\overleftrightarrow{AC} \cap \ell = \{B\}$ we know that $\operatorname{int}(\overrightarrow{BA}) \cap \ell = \emptyset$.

Thus from (a), all of $\operatorname{int}(\overrightarrow{BA})$ lies on the same side of ℓ.

Since $\operatorname{int}(\overline{BA}) \subseteq \operatorname{int}(\overrightarrow{BA})$ we know $\operatorname{int}(\overline{BA})$ and $\operatorname{int}(\overrightarrow{BA})$ lie on the same side of ℓ.



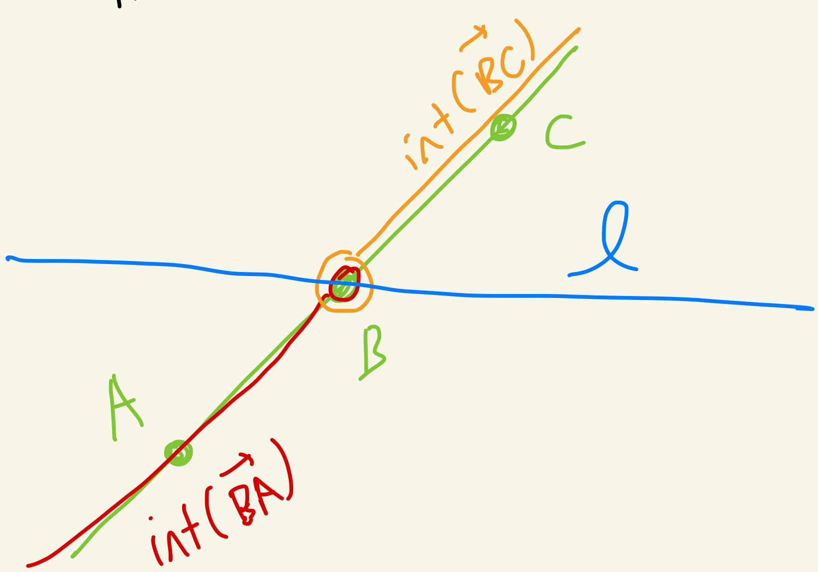
What about $\text{int}(\overrightarrow{BC})$ and $\text{int}(\overrightarrow{BA})$?

Since $\overleftrightarrow{AC} \cap \ell = \{B\} \neq \emptyset$ we know that A and C lie on opposite sides of ℓ.

We know all of $\text{int}(\overrightarrow{BA})$ lies on one side of ℓ.

A similar argument shows that all of $\text{int}(\overrightarrow{BC})$ lies on the same side of ℓ.

Since $A \in \text{int}(\overrightarrow{BA})$ and $C \in \text{int}(\overrightarrow{BC})$ this gives that $\text{int}(\overrightarrow{BA})$ and $\text{int}(\overrightarrow{BC})$ lie on opposite sides of ℓ.



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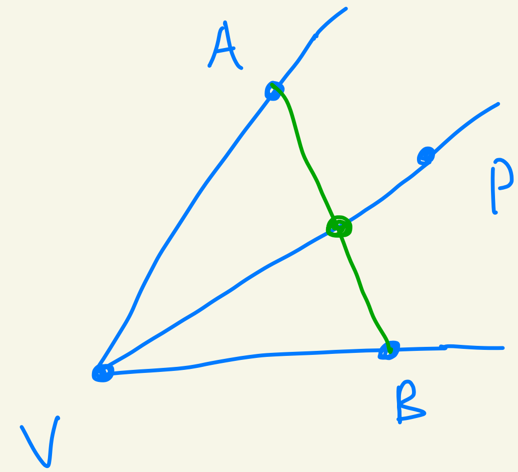
② Suppose $\angle AVB$ is an angle and B and P are on the same side of $\overleftrightarrow{VA}$.

($\Rightarrow$) Suppose $P \in \operatorname{int}(\angle AVB)$.

Then, $\overrightarrow{VP} \cap \overline{AB} \neq \phi$ by the crossbar theorem.

So, $\overleftrightarrow{VP} \cap \overline{AB} \neq \phi$.

So, A and B are on opposite sides of $\overleftrightarrow{VP}$.

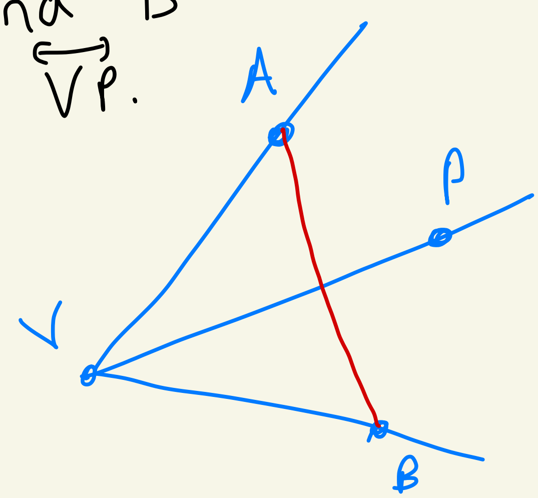


($\Leftarrow$) Now suppose that A and B are on opposite sides of $\overleftrightarrow{VP}$.

By the PSP axiom,

$\overline{AB} \cap \overleftrightarrow{VP} \neq \phi$

Since $A, B \notin \overleftrightarrow{VP}$ we get $\boxed{\operatorname{int}(\overline{AB}) \cap \overleftrightarrow{VP} \neq \phi.}$ (*)



Since $\overleftrightarrow{AB} \cap \overleftrightarrow{VA} = \{A\}$ we know

$\operatorname{int}(\overrightarrow{AB}) \cap \overleftrightarrow{VA} = \phi$.

Thus from problem 1, all

of $\text{int}(\overrightarrow{AB})$ lies on the same side of $\overleftrightarrow{VA}$.

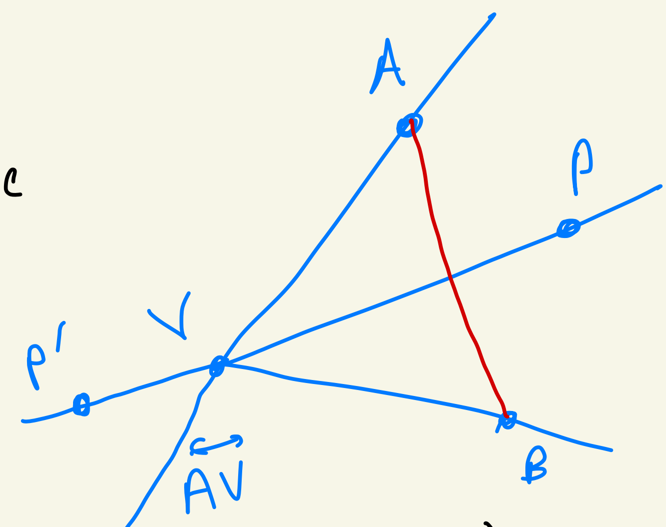
Since $B \in \text{int}(\overrightarrow{AB})$, and by assumption B and P are on the same side of $\overleftrightarrow{VA}$, we get that $\text{int}(\overrightarrow{AB})$ and P lie on the same side of $\overleftrightarrow{VA}$.

Since $\text{int}(\overline{AB}) \subseteq \text{int}(\overrightarrow{AB})$ we get that $\text{int}(\overline{AB})$ and P lie on the same side of $\overleftrightarrow{VA}$.

Note that $\text{int}(\overrightarrow{VP})$ lies on one side of $\overleftrightarrow{VA}$ by problem 1.

Let $P'-V-P$.

Then, $\text{int}(\overrightarrow{VP})$ is on one side of $\overleftrightarrow{AV}$ and $\text{int}(\overrightarrow{VP'})$ is on the other side.



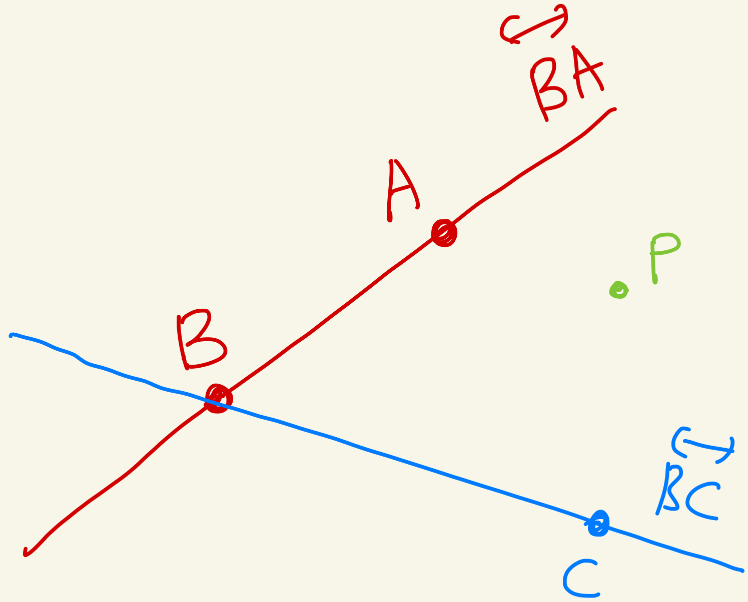
Since $P \in \text{int}(\overrightarrow{VP})$ we know $\text{int}(\overrightarrow{VP})$ and $\text{int}(\overline{AB})$ are on the same side of $\overleftrightarrow{AV}$.

Combine this with (*) gives $\overrightarrow{VP} \cap \text{int}(\overline{AB}) \neq \phi$.

By the previous hw problem, $P \in \text{int}(\angle AVB)$.

$\boxed{///}$

3. Suppose that A, B, C are non-collinear in a Pasch geometry.



By the definition of $\text{int}(\angle ABC)$, we know that

$$P \in \text{int}(\angle ABC)$$

iff

both P and C lie on the same side of $\overleftrightarrow{BA}$ and P and A lie on the same side of $\overleftrightarrow{BC}$.

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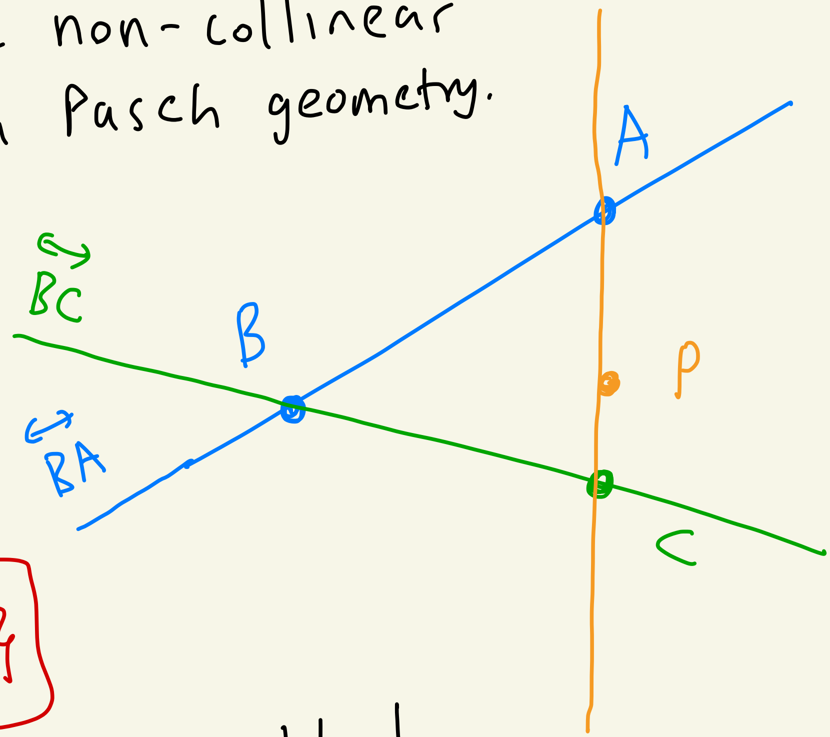
④ Suppose A, B, C are non-collinear and $A-P-C$ in a Pasch geometry.



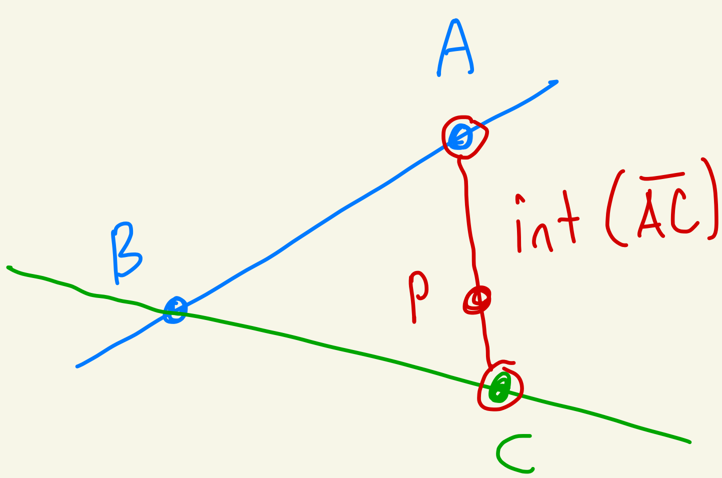
Since $A-P-C$ we know that $P \in \operatorname{int}(\overline{AC})$

[Since $\operatorname{int}(\overline{AC}) = \overline{AC} - \{A, C\}$]

Thus we only need to show that

$$\operatorname{int}(\overline{AC}) \subseteq \operatorname{int}(\angle ABC)$$

Let's show that all of $\operatorname{int}(\overline{AC})$ lies on the same side of $\overleftrightarrow{BC}$.

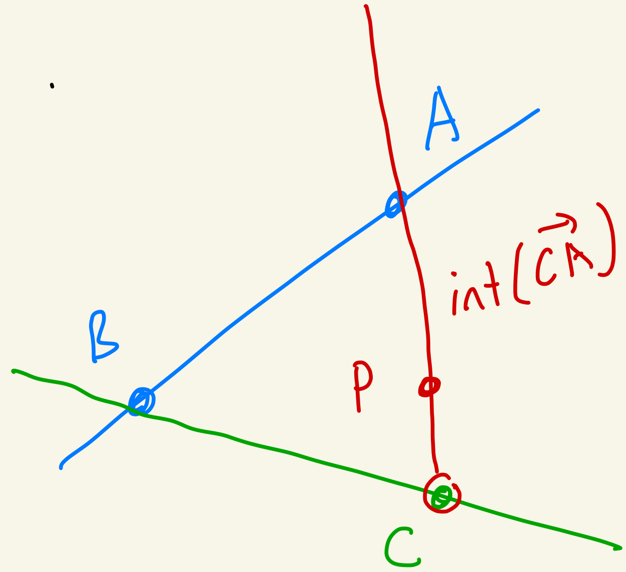


Because A, B, C are non-collinear we know that $\overleftrightarrow{AC} \neq \overleftrightarrow{BC}$.

Thus, $\overleftrightarrow{AC} \cap \overleftrightarrow{BC} = \{C\}$.

Since $\operatorname{int}(\overrightarrow{CA}) \subseteq \overleftrightarrow{AC}$ and $C \notin \operatorname{int}(\overrightarrow{CA})$

this gives $\operatorname{int}(\overrightarrow{CA}) \cap \overleftrightarrow{BC} = \phi$.



Thus, by problem 1(a) all of $\operatorname{int}(\overrightarrow{CA})$ lies on the same side of $\overleftrightarrow{BC}$.

Since $A \in \operatorname{int}(\overrightarrow{CA})$ we know all of $\operatorname{int}(\overrightarrow{CA})$ lies on the same side of $\overleftrightarrow{BC}$ as A does.

A similar argument shows that all of $\operatorname{int}(\overrightarrow{AC})$ lies on the same side of $\overleftrightarrow{BA}$ as C.

Since $\operatorname{int}(\overrightarrow{AC}) \cap \operatorname{int}(\overrightarrow{CA}) = \operatorname{int}(\overline{AC})$ the above shows that (i) all of $\operatorname{int}(\overline{AC})$ lies on the same side of $\overleftrightarrow{BC}$ as A does, and (ii) all of $\operatorname{int}(\overline{AC})$ lies on the same side of $\overleftrightarrow{BA}$ as C does.

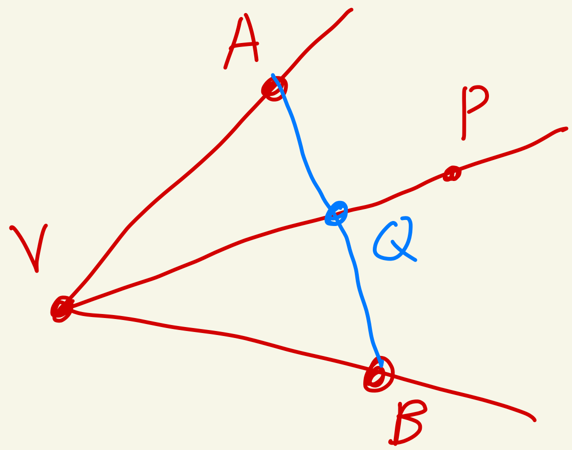
Thus, by the def of $\operatorname{int}(\angle ABC)$ we get that $\operatorname{int}(\overline{AC}) \subseteq \operatorname{int}(\angle ABC)$.

$\blacksquare$

5 Suppose $\angle AVB$ is an angle and $\overrightarrow{VP} \cap \text{int}(\overline{AB}) \neq \emptyset$.

Suppose $\overrightarrow{VP} \cap \text{int}(\overline{AB}) = \{Q\}$. ← possibly $P = Q$

Then, $Q \in \text{int}(\overline{AB})$.



So, $A - Q - B$.

Thus, A and Q are on the same side of $\overleftrightarrow{VB}$. $(*)$

Since $\overrightarrow{VP} \cap \overleftrightarrow{VB} = \{V\}$ we know $\text{int}(\overrightarrow{VP}) \cap \overleftrightarrow{VB} = \emptyset$.

Thus, since $\text{int}(\overrightarrow{VP})$ is convex we know $\overline{PQ} \subseteq \text{int}(\overrightarrow{VP})$ and so $\overline{PQ} \cap \overleftrightarrow{VB} = \emptyset$.

Thus, P and Q are on the same side of $\overleftrightarrow{VB}$. $(**)$

By $(*)$ and $(**)$ A and P are on the same side of $\overleftrightarrow{VB}$. (i)

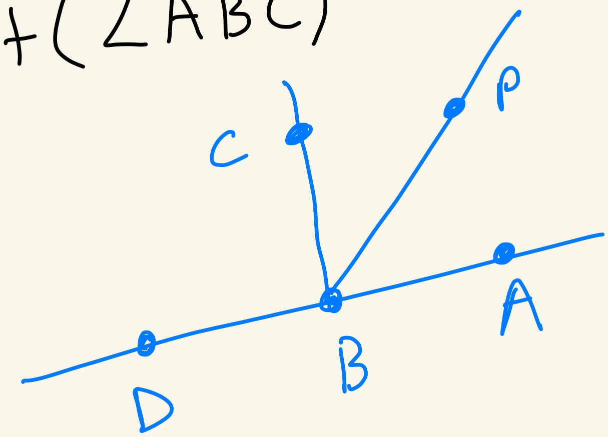
A similar argument will give P, Q, B are on the same side of $\overleftrightarrow{VA}$.

Thus, P and B are on the same side of $\overleftrightarrow{VA}$ (ii)

By (i) and (ii) we get $P \in \operatorname{int}(\angle AVB)$.

$\blacksquare$

⑥ Let A, B, C, D, P be points in a Pasch geometry. Suppose that A, B, C are non-collinear and that $A-B-D$.



($\Rightarrow$) Suppose that $P \in int(\angle ABC)$

Then, P and C are on the same side of $\overleftrightarrow{AB}$

Since $\overleftrightarrow{AB} = \overleftrightarrow{DB}$ this means P and C are on the same side of $\overleftrightarrow{DB}$.

FINISH

($\Leftarrow$) Suppose that $C \in \mathrm{int}(DBP)$.

(7) Let A, B, C be non-collinear points in a Pasch geometry.

We must show that $\text{int}(\Delta ABC)$ is convex.

Let H_1 be the side of $\overleftrightarrow{AB}$ that contains C.

Let H_2 be the side of $\overleftrightarrow{BC}$ that contains A

Let H_3 be the side of $\overleftrightarrow{CA}$ that contains B.

Since a Pasch geometry satisfies the PSA axiom we have that H_1, H_2, H_3 are convex.

By Homework 7 we know then that $H_1 \cap H_2 \cap H_3$ is a convex set.

Hence,

$$\text{int}(\Delta ABC) = H_1 \cap H_2 \cap H_3$$

is convex. $\blacksquare$